Flexible Solutions to Linear Programs under Uncertainty: Inequality Constraints

In most industrial applications the linear model used for optimization by linear programming involves significant uncertainties and inaccuracies in the model parameters. This paper presents a framework which allows uncertainties in the matrix elements of the linear program to be taken into account without requiring detailed knowledge of the statistical characteristics of these uncertainties. Three cases for the inequality constrained problem are considered: independent variations in the array elements, column dependent variations, and row dependent variations. In each case the problem can still be solved as a linear program. In the first two cases, the problem size is doubled, while for the row dependent case a finitely terminating cutting plane algorithm is constructed.

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SCOPE

Linear programming is now widely used to schedule production, allocate resources, design blending operations, as well as in many other applications requiring the determination of optimal operating policies for complex interacting systems. The construction of the necessarily linear model of the system upon which the technique operates often requires the use of linear approximations to describe intrinsically nonlinear phenomena and always involves uncertainty in the estimates of the model parameters. For instance, octane number does not blend linearly and model parameters such as processing costs, production yields, raw materials properties and availabilities, as well as product compositions and demands are all subject to variation. As a consequence, the optimal solution obtained via the LP model cannot be accepted as a firm policy and will always require some adjustment to account for these inaccuracies as well as any real changes in the process environment.

Analytic approaches to dealing with uncertainties in the model parameters of a linear program have proceeded along two directions: sensitivity analysis (Beightler and Wilde, 1965), the simpler of the two approaches, considers one-at-a-time variations in the cost coefficients and constraint right-hand sides and calculates bounds on the variations at which a change in the optimal basis will be brought about. Simultaneous variation of several parameters as well as uncertainties in the matrix elements cannot be accommodated. The more sophisticated probabilistic programming analysis (Vajda, 1972) takes into account the detailed statistical properties of the parameter variations but has primarily been developed for linear programs with stochastic right-hand sides. The theory can in principle be extended to deal with stochastic matrix elements, but the computations then become highly nonlinear and complex. Moreover, from a practical point of view, the probabilistic programming approach is especially unsatisfactory in the case of uncertainties in the matrix elements because usually the required statistical distributions and their parameters are not known and the variation in the elements may not be independent. Typically, the model builder will know a good, that is, mean value for each matrix element, the range within which variation in each element may be expected, and any algebraic relationship between the array elements. For example, a selection of the array elements may represent compositions (as in the standard blending problem), and hence these must sum to some normalizing factors; or else, a set of array elements may represent yields of each product per unit of raw material (as in a production scheduling problem), in which case the conservation law will impose a linear constraint on the total yield obtained from a given raw material.

Finally, the probabilistic programming approaches are generally too conservative in the sense that they assume that once a policy is decided upon it cannot be changed in the future. Actually, policy adjustments are always and continually made to meet changes or correct unsatisfactory operations. It is desirable that the planner/engineer be able to anticipate the required changes and, thus, to provide in advance the flexibility which will permit future adjustments to be made at minimum cost. These considerations provide the motivation for the model and procedures discussed herein.

CONCLUSIONS AND SIGNIFICANCE

This paper presents a framework which allows uncertainty in the matrix elements of a linear program to be taken into account. Specifically, three cases for the linear inequality constrained problem are considered: independent variations in the array elements; row dependent variations, that is, the variations of the elements in given rows are algebraically dependent; and column dependent variations. The resulting feasible region for each case is characterized and computational procedures are detailed.

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In all three cases, the problem can still be solved as a linear program. In the independent and column dependent cases, the problem size is doubled, while in the row dependent case an iterative but finitely terminating cutting plane procedure is necessitated. In applications involving large linear programming models all three cases may occur simultaneously, in which event the use of the iterative procedure will always be required. As a result of these calculations, the engineer/planner will obtain a balance between the optimal operating policy based on current best information and the optimal amount of flexibility which will be required in the policy in order to be able to accommodate it to model uncertainties.

The literature on linear programming provides two main approaches for evaluating and accounting for uncertainty in the model parameters: sensitivity analysis and probabilistic programming. The former is too simple to account for uncertainties in more than one parameter at a time, while the latter requires detailed statistical properties of the parameter variations and has evolved around two types of models:

i) the two-stage program
ii) the chance constrained program

The two-stage programming approach (Dantzig, 1955; Walkup and Wets, 1970) replaces the problem of

$$\begin{aligned} &\text{Minimizing} \quad cx \\ &\text{Subject to} \quad Ax = b \\ &\qquad\qquad\quad x \geqq 0 \end{aligned}$$

where b is random with known distribution, with that of

$$\begin{aligned} &\text{Minimizing} \quad cx + E(dy) \\ &\text{Subject to} \quad Ax + By = b \\ &\qquad\qquad\quad x, y \geqq 0 \end{aligned}$$

where E is the expected value of dy over values of the random variable y or, more directly, b since

$$By = b - Ax$$

This model thus takes into account the uncertainties in parameters b by choosing a policy X which not only minimizes the cost cx, but also the expected cost of any future violations of the constraints. Even in the simplest case, solution of the two-stage problem is attained only via a nonlinear convex program. The case of random array elements a_{ij} has been considered (Walkup and Wets, 1970) but leads to an impractically difficult nonlinear programming problem.

The second model, the chance constrained program (Charnes and Cooper, 1959, 1963), replaces the conventional linear program with

$$\begin{aligned} &\text{Minimize} \quad cx \\ &\text{Subject to} \quad \text{Prob}\left(\sum_j a_{ij} x_j \geqq b_i\right) \geqq \gamma_i, \quad i = 1, M \\ &\qquad\qquad\quad x \geqq 0 \end{aligned} \tag{3}$$

where Prob ( ) indicates the probability that the event within the parenthesis occurs and the γ_i are given nonzero probabilities. This model thus allows for the possibility that the constraints may in unusual circumstances be violated and only insists that they be satisfied most of the time. It can be shown that a chance constrained of the above form can under reasonable hypothesis be replaced by a linear deterministic equivalent providing the distribution of the b_i are known. The case of random a_{ij} has been considered and a linear deterministic equivalent derived for a_{ij} independent and normally distributed (Vajda, 1970; Charnes and Cooper, 1959, 1963). Other probability distributions lead to complex nonlinear deterministic equivalents (Charnes et al., 1966).

In addition to computational difficulties, the above approaches are unsatisfactory in the case of uncertainties in the a_{ij} because usually the statistical distributions and their parameters are not known and the variations in the a_{ij} are often not independent. Moreover, both probabilistic programming approaches, especially the two-stage programs, are too conservative since they assume that once a policy is decided upon it cannot be changed. Hence, they calculate a policy which has, in some sense, little chance of being infeasible whatever the errors in the model. Yet in practice, policies can and are changed providing provisions for adjustment have been made in advance. The strategy adopted in this paper is that an operating policy should be calculated based on the model using average or best values of the parameters. However, to ensure that in the least favorable event a revised feasible policy can always be implemented, a least cost flexibility region is also determined. The problem which is posed is thus, given the uncertainties in the model, what is the optimal policy based on current best information, and what is the allowance for flexibility which must be purchased in order to make that policy safe.

LINEAR PROGRAMS WITH FLEXIBILITY

Suppose each coefficient a_{ij} and b_i has a known range of variation

$$\begin{aligned} -\alpha_{ij} \leqq \delta a_{ij} \leqq \alpha_{ij} \quad i = 1, \ldots, M \\ -\beta_i \leqq \delta b_i \leqq \beta_i \quad j = 1, \ldots, N \end{aligned} \tag{4}$$

that is, the variations in each coefficient are uniformly distributed on a known interval.

Moreover, suppose that the coefficients of certain specified rows and columns of the matrix A are algebraically dependent. Thus, there are known coefficients η_{ij}, μ_{ij} such that

$$\begin{aligned} \sum_i \eta_{ij} a_{ij} = \eta_{0j} \quad j \in C \\ \sum_j \mu_{ij} a_{ij} = \mu_{i0} \quad i \in R \end{aligned}$$

where C is the set of column indices corresponding to columns whose coefficients are dependent and R is the set of indices of the rows whose coefficients are dependent. Equivalently, it follows that

$$\begin{aligned} \sum_i \eta_{ij}\, \delta a_{ij} = 0 \quad j \in C \\ \sum_j \mu_{ij}\, \delta a_{ij} = 0 \quad i \in R \end{aligned} \tag{5}$$

For each choice of variations $\delta A = (\delta a_{ij})$, $\delta b = (\delta b_i)$ satisfying conditions (4) and (5), the corresponding perturbed constraint set will be

$$\sum_j (a_{ij} + \delta a_{ij}) X_j \leqq b_i + \delta b_i \quad i = 1, \ldots, M$$

Each such constraint set will define a perturbed feasible region

$$F(\delta A, \delta b) = \{y : \sum_j (a_{ij} + \delta a_{ij})\, y_j \leq b_i + \delta b_i,$$

$$i = 1, \ldots, M \quad \text{and} \quad y \geq 0\}$$

In this notation, the set $F(0)$ will represent $F(\delta A, \delta b)$ with $\delta A = 0$, $\delta b = 0$, that is, the original constraint set using the best estimates a_{ij}, b_i.

Now, let P denote the set of all variations $(\delta A, \delta b)$ satisfying (4) and (5), then the problem with flexibility is

$$\begin{aligned} &\text{Minimize} \quad cx + c^+z + c^-z^- \\ &\text{Subject to} \quad x \in F(0) \\ &\qquad\qquad x + z^+ - z^- \in F(\delta A, \delta b) \end{aligned} \tag{6}$$

where the particular choice of perturbation $(\delta A, \delta b)$ can be any of the perturbations from the set P.

In the above formulation, variables z^+ and z^- represent the corrections, positive and negative, which may have to be applied to the policy x, which satisfies the best estimate model, if the coefficients of the model actually are perturbed by $(\delta A, \delta b)$. The cost coefficients c^+ and c^- are the costs of flexibility. Note that these should not be the costs of making the corrections themselves (which are costs that may or may not be incurred in the future depending upon chance) but rather the costs payable now in order to make provisions for possible future corrections. For instance, they might be the cost of an option to purchase z^+ additional or z^- less units in a contract written for the purchase of X units. Thus the costs c_i^+ and c_i^- will usually be less than the corresponding c_i, will usually differ in magnitude, and may, of course, be zero.

As a consequence of the assumption of uniform distribution of the array coefficients, it follows that each choice of $(\delta A, \delta b)$ in the above problem is equally likely and, hence, all possible sets $F(\delta A, \delta b)$ must be considered. In general, the coefficients could follow other statistical distributions and thus would lead to joint distributions of $(\delta A, \delta b)$, requiring a more complex analysis than that advanced here. In the absence of further statistical information, the uniform distribution is an adequate conservative choice. Let

$$F^* = \bigcap_{(\delta A, \delta b) \in P} F(\delta A, \delta b) \text{ and assume that } F^* \neq 0.$$

Then, problem (6) can be stated as

$$\begin{aligned} &\text{Minimize} \quad cx + c^+ z^+ + c^- z^- \\ &\text{Subject to} \quad x \in F(0) \\ &\qquad\qquad x + z^+ - z^- \in F^* \end{aligned} \tag{7}$$

In the following section, the set F^* called the *permanently feasible set* will be characterized.

PERMANENTLY FEASIBLE SET

Since each set $F(\delta A, \delta b)$ is defined by a collection of linear inequalities, it is convex and polyhedral. Consequently, F^* will be convex and $F^* \subset F(\delta A, \delta b)$. Fortunately, F^* is also polyhedral and it is this property which permits (7) to be solved as a linear programming problem. For convenience in exposition, we will consider three separate cases although in practice the three may occur simultaneously:

1. All δa_{ij}, δb_i independent
2. Column dependence only (that is, $R = \phi$)
3. Row dependence only (that is, $C = \phi$)

Case 1: Independent Perturbation

Proposition 1. If the perturbations are independent, then F^* will be defined by

$$\{y : \sum (a_{ij} + \alpha_{ij})\, y_j \leq b_i - \beta_i,$$

$$i = 1, \ldots, M \quad \text{and} \quad y_j \geq 0$$

Proof: Let T indicate the above set, and suppose y is in T. Then for any choice $(\delta A, \delta b)$ satisfying (4) (including $\delta A = 0$, $\delta b = 0$), the following inequalities must hold:

$$\sum_j (a_{ij} + \delta a_{ij})\, y_j \leq \sum (a_{ij} + \alpha_{ij})\, y_j \leq b_i - \beta_i \leq b_i + \delta b_i,$$

$$i = 1, \ldots, M$$

Consequently, $T \subset F(\delta A, \delta b)$ and of course $T = F(\alpha, \beta)$. Therefore, $T \cap F(\delta A, \delta b) \equiv F^*$.

Note that the above set is polyhedral.

Case 2: Column-Dependent Perturbations

Proposition 2. If the perturbations are column dependent, then F^* will be defined by

$$\{y : \sum (a_{ij} + \delta^* a_{ij})\, y_j \leq b_i - \beta_i,$$

$$i = 1, \ldots, M \quad \text{and} \quad y_j \geq 0\}$$

where for each $i = 1, \ldots, M$ and $j = 1, \ldots, N$

$$\delta^* a_{ij} = \begin{cases} \text{MIN}\left\{ \alpha_{ij}, \ \sum_{\substack{k=1 \\ k \neq i}}^{M} \left| \dfrac{\eta_{kj}\, \alpha_{kj}}{\eta_{ij}} \right| \right\} & \text{if} \quad \eta_{ij} \neq 0 \\ \alpha_{ij} \quad \text{otherwise} & \end{cases}$$

Proof: Let T indicate the above set and suppose that for each separate constraint i, $i = 1, \ldots, M$, each perturbation δa_{ij} of the coefficients of that constraint are chosen such as to

$$\begin{aligned} &\text{Maximize} \quad \delta a_{ij} \\ &\text{Subject to} \quad \sum_k \eta_{kj}\, \delta a_{kj} = 0 \quad j \in C \\ &\qquad\qquad -\alpha_{ij} \leq \delta a_{ij} \leq \alpha_{ij} \end{aligned}$$

Let $\delta \bar{a}_{ij}$ correspond to the above choice of δa_{ij} and $\delta \bar{A} = (\delta \bar{a}_{ij})$. Clearly, if $y \in T$ then, for all other $(\delta A, \delta b) \in P$ the following inequalities must hold for each $i = 1, \ldots, M$:

$$\sum_j (a_{ij} + \delta a_{ij})\, y_j \leq \text{Max} \sum_j (a_{ij} + \delta a_{ij})\, y_j$$

$$= \sum_j (a_{ij} + \delta \bar{a}_{ij})\, y_j \leq b_i - \beta_i \leq b_i + \delta b_i \tag{8}$$

Consequently, $T \subset F(\delta A, \delta b)$ and, again, $T = F(\delta \bar{A}, \beta)$. Therefore, $T = \bigcap_{(\delta A, \delta b) \in P} F(\delta A, \delta b) \equiv F^*$.

It remains to verify the computations involved in the above maximization subproblem. First of all, if $\eta_{ij} = 0$, then δa_{ij} is unconstrained and can be set at its bound α_{ij}. On the other hand, if $\eta_{ij} \neq 0$, then it must satisfy

$$\delta a_{ij} = -\frac{1}{\eta_{ij}} \sum_{k \neq i} \eta_{kj}\, \delta a_{kj}$$

The maximum value of the right-hand side is attained if, for each k, $\delta a_{kj} = -\alpha_{kj}$ if η_{kj}/η_{ij} is positive, and $\delta a_{kj} = +\alpha_{kj}$ if (η_{kj}/η_{ij}) is negative. Hence, the absolute value in Proposition 2 is justified and Case 2 is verified.

Note that the set F^* is again polyhedral and that F^* for case 1 (independent perturbations) is contained in F^* for case 2. Note also that dependence in the variation of the b_i can be handled exactly as column dependence in the δa_{ij} if it is recognized that each δb_i should individually be minimized.

Case 3: Row-Dependent Perturbations

For each constraint i with row dependence, let

$$S_i = \{t : t \text{ is a basic feasible solution of } \sum_j \mu_{ij} t_j = 0 \text{ and } -\alpha_{ij} \leq t_j \leq \alpha_{ij}, \quad j = 1, \ldots, N\}$$

For each $t \in S_i$, let

$$T_i(t) = \{y : \sum_j (a_{ij} + t_j) y_j \leq b_i, \quad y_j \geq 0\}$$

and, define

$$T_i = \bigcap_{t \in S_i} T_i(t)$$

Finally, for all constraints i with no row dependence, let

$$T_i = \{y : \sum_j (a_{ij} + \alpha_{ij}) y_j \leq b_i - \beta_i, \quad y_j \geq 0\}$$

Proposition 3. If the perturbations are row dependent, then $F^* = \bigcap_{i=1}^{M} T_i$ and is polyhedral.

Proof: First of all, note that since the hypercube, $-\alpha_{ij} \leq t_j \leq \alpha_{ij}$, $j = 1, \ldots, N$ has exactly 2^N basic feasible solutions, the restriction to the hyperplane $\sum_j \mu_{ij} t_j = 0$ will have at most 2^{N-1} basic feasible solutions. Consequently, each set T_i is polyhedral since it is composed of the intersection of only a finite number of polyhedral sets. Thus, F^* itself must also be polyhedral.

Next, suppose $y \in \bigcap_{i=1}^{M} T_i$. The case $i \notin R$ follows from Proposition 1. Consequently, consider only the case $i \in R$; that is, the constraint i has row-dependent perturbations. Clearly, if $\delta \bar{a}_i = (\delta \bar{a}_{ij})$ is the solution to, maximize

$$\sum_j \delta a_{ij} y_j$$

$$\text{Subject to} \quad \sum_j \mu_{ij} \delta a_{ij} = 0 \qquad (9)$$

$$-\alpha_{ij} \leq \delta a_{ij} \leq \alpha_{ij}$$

then, inequalities (8) will again hold true and Proposition 3 will be proven. To see that this is the case, note that all solutions to subproblem (9) must be basic feasible solutions and thus are contained in S_i. But since y is in T_i, it also will be in the $T_i(\delta \bar{a}_i)$ corresponding to the maximizing solution $\delta \bar{a}_i$ of (9). The rest follows as in Proposition 2.

Note that although F^* is polyhedral and thus can be defined by a finite collection of linear inequalities, this collection may consist of up to $M2^{N-1}$ members. Complete enumeration may thus prove quite impractical.

In the next section, computational approaches towards solving problem (7) for each of the above three cases are presented.

SOLUTION PROCEDURES

The solution of problem (7) in the case of independent or column dependent variations (case 1 and 2, respectively) merely requires the solution of a linear programming problem of twice the size of the original, best estimate model. In both of these cases problem (7) may be stated as

$$\text{Minimize} \quad cx + c^+ z^+ + c^- z^-$$

$$\text{Subject to} \quad \sum a_{ij} x_j \leq b_i \qquad i = 1, \ldots, M \qquad (10)$$

$$\sum (a_{ij} + \delta \bar{a}_{ij})(x_j + z_j^+ - z_j^-) \leq b_i - \beta_i, \quad i = 1, \ldots, M$$

$$x_j, z_j^+, z_j^- \text{ all } \geq 0 \text{ and } x_j - z_j^- \geq 0 \quad j = 1, \ldots, N$$

where the inequality $x_j + z_j^+ - z_j^- \geq 0$ has been replaced by $x_j^+ - z_j^- \geq 0$ because, as is well known, z_j^+ and z_j^- can not both be greater than zero. In the case of independent variations, $\delta \bar{a}_{ij} = \alpha_{ij}$; while , in the case of column dependent variations, $\delta \bar{a}_{ij} = \delta a^*_{ij}$, where δa^*_{ij} is calculated as defined in Proposition 2.

The computations for case 3, the row dependent case, are somewhat more complex because of the nature of its permanenty feasible region. It should be pointed out, however, that since F^* of the independent case is always contained within those of the dependent cases, a conservative solution to cases 2 and 3 can always be obtained by solving case 1.

Because of the large number of linear inequalities which may be required in order to solve case 3 in a once-through manner as is done for the other two cases, an iterative procedure is deemed more appropriate. The algorithm to be described next may be viewed as a cutting plane algorithm (Zangwill, 1969) in which the successive cuts are generated by solving subproblem (9). The intermediate problems are of course all linear programming problems. Our procedure differs from the usual cutting plane algorithm in that it does not rely on convergence in the limit but in fact has finite convergence. We begin by outlining the overall cutting plane procedure itself; then, specify the procedure for generating cuts; and, finally, conclude with some remarks on convergence.

Let (A_2, b_2) denote the rows (A, b) which have row dependent variations and let (A_1, b_1) be those with independent variations.

ALGORITHM FOR ROW DEPENDENCE

Solve the original problem (1) to obtain a solution x^1. Set $z^+ = z^- = 0$.

For iterations $n = 1, 2, 3, \ldots$

I. Generate the cut at $(x + z^+ - z^-)^n$: solve subproblem (9) to obtain variation, δA_2^n

II. Test for termination: if $(x + z^+ - z^-)^n$ satisfies

$$(A_2 + \delta A_2^n)(x + z^+ - z^-) \leq b_2 - \beta_2$$

within some specified tolerance, then terminate the iterations. Otherwise, continue with (III)

III. Solve the linear program,

Subject to $Ax \leq b$

$$(A_1 + \alpha_1)(x + z^+ - z^-) \leq b_1 - \beta_1$$

$$(A_2 + \delta A_2^k)(x + z^+ - z^-) \leq b_2 - \beta_2 \qquad k = 1, \ldots, n$$

$$x, z^+, z^- \geq 0 \quad \text{and} \quad x - z^- \geq 0$$

Denote the solutions by x^{n+1} and $(x + z^+ - z^-)^{n+1}$; set $n = n + 1$, and go to step (I).

The number of inequalities which need to be retained in step (III) may be reduced somewhat by dropping any of the new inequalities which test feasible in step (II).

The following is an elaboration of step (I):

Cut Generating Algorithm

For each i, let $J_i = \{j : \mu_{ij} < 0\}$ and for each $j \in J_i$, set $\mu_{ij} = -\mu_{ij}$, $x_j + z_j^+ - z_j^- = -(x_j + z_j^+ - z_j^-)$. Then, for each constraint i with row dependence:

1. Compute $\theta_i = \frac{1}{2} \sum_j \mu_{ij} \alpha_{ij}$ and set $\gamma = \text{sum} = 0$.
2. Order index j according to decreasing values of

TABLE 1. CRUDE YIELDS AND COST

Product yield, vol. %	Crude type 1	2	3	4 Fuel	4 Lube	Product on order, M barrels/week
Gasoline	0.6	0.5	0.3	0.4	0.4	170
Heating oil	0.2	0.2	0.3	0.3	0.1	85
Lube oil	0	0	0	0	0.2	20
Jet fuel	0.1	0.2	0.3	0.2	0.2	85
Loss	0.1	0.1	0.1	0.1	0.1	—
Crude available, M barrels/week	100	100	100	200		—
Contribution to profit, \$/M barrels crude	100	200	70	150	250	—

$(x_j + z_j^+ - z_j^-)/\mu_{ij}$ $(\mu_{ij} \neq 0)$. Let $\{m_k\}$ be the sequence of j's. Set $k = 1$ and compute

(a) $\mathrm{SUM} = \mathrm{SUM} + \mu_{im_k}\,\alpha_{im_k}$

If $\mathrm{SUM} < \theta_i$, set $k = k + 1$, $\gamma = \mathrm{SUM}$ and continue with (a)

If $\mathrm{SUM} \geq \theta_i$, set $\Delta = (\theta_i - \gamma)/\mu_{im_k}\,\alpha_{im_k}$ and go to (b)

(b) Set $\delta a_{im_l} = \alpha_{im_l}$ $\quad l = 1, \ldots, k - 1$

$\delta a_{im_k} = \alpha_{im_k} \quad (2\Delta - 1)$

$\delta a_{im_l} = -\alpha_{im_l}$, otherwise

3. Set $\delta a_{ij} = -\delta a_{ij}$ for all $j \in J_i$, increment $i = i + 1$, and continue with step 1.

The above procedure solves subproblem (9) using the standard algorithm for the knapsack problem (Beightler and Wilde, 1967). To see this, observe that with the sign change $\mu_{ij} = -\mu_{ij}$ and $y_j = -y_j$, for each μ_{ij} which is negative, together with the change of variable

$$t_j = \frac{\delta a_{ij} + \alpha_{ij}}{2\alpha_{ij}}$$

subproblem (9) becomes

$$\text{Maximize} \quad \sum_j (\alpha_{ij}\, y_j) t_j$$

$$\text{Subject to} \quad \sum_j (\mu_{ij}\,\alpha_{ij})\, t_j = \tfrac{1}{2} \sum \mu_{ij}\,\alpha_{ij}$$

$$0 \leq t_j \leq 1$$

where each coefficient of the equality as well as its right-hand side is positive. This is a continuous variable knapsack problem, the solution to which requires merely the ordering of the t_j according to decreasing y_j/μ_{ij}. Let t_{m_k} be such an ordering of the t_j's. The solution is obtained by setting successive t_{m_k} equal to 1 until the constraint right-hand side is exceeded. All other t_{m_k} are set equal to zero, while the t_{m_k} which forces the constraint to be exceeded is assigned an appropriate fractional value. The procedure outlined above is a direct implementation of these steps.

Proposition 4: Providing $F^* \neq \phi$, the above algorithm will terminate at a point $x + z^+ - z^- \in F^*$ in a finite number of steps.

Proof: From the statement of the algorithm, the iterations will terminate at $(x + z^+ - z^-)^n$ only if the new constraints

$$(A_2 + \delta A_2^n)(x + z^+ - z^-) \leq b_2 - \beta_2$$

are satisfied. But if this is the case, then since A_2^n is the optimal solution to subproblems (9), it follows that for each $i \in R$

$$\sum_j (a_{ij} + \delta a_{ij})(x + z^+ - z^-)_j^n \leq \sum_j (a_{ij} + \delta a_{ij}^n)(x + z^+ - z^-)_j \leq b_i - \beta_i$$

for all possible δa_{ij}. Since this inequality holds for all δa_{ij}, it must in particular hold true for all δa_i in S_i. Consequently, $(x + z^+ - z^-)^n$ must be contained in each T_i and, therefore, in F^*.

On the other hand, if $(x + z^+ - z^-)^n$ does not satisfy the new constraints, then the new variation δA_2^n must have introduced at least one new basic solution δa_i belonging to one of the sets S_i. Consequently, at least one new half plane T_i (δa_i) of the set of T_i's whose intersection defines F^* will have been generated. Since there is only a finite number of the T_i (δa_i), it follows that after a finite number of iterations no new T_i (δa_i) will be generated and thus an iterate reached which will pass the feasibility test in step (II) of the algorithm.

Although a statement of finite termination is reassuring, it is not anticipated that all of the half-planes defining F^* will have to be accumulated in order to achieve termination. Instead, because of the form of subproblem (9) it is expected that only the locally significant half-planes will be generated and that thus only a small number of iterations will be required.

EXAMPLE

As an illustration of the techniques presented above, we consider the refinery scheduling problem discussed in (Beightler and Wilde, 1965).

A refinery purchases four types of crudes which are sent either through a fuels or a lube process to produce four products: gasoline, heating oil, jet fuel, and lube oil. The product yields, product requirements, crude availabilities, and profit contributions for each crude are tabulated in Table 1. If x_i, $i = 1, \ldots, 4$ is the amount of crude i, in thousands of barrels per week, purchased and utilized in the fuels process and x_5 is the M barrels/week of crude 4 purchased and utilized in the lube process, then the following linear program can be formulated:

Maximize

$$100\, x_1 + 200\, x_2 + 70\, x_3 + 150\, x_4 + 250\, x_5$$

Subject to

$$0.6\, x_1 + 0.5\, x_2 + 0.3\, x_3 + 0.4\, x_4 + 0.4\, x_5 \leq 170$$

$$0.2\, x_1 + 0.2\, x_2 + 0.3\, x_3 + 0.3\, x_4 + 0.1\, x_5 \leq 85$$

$$0.1\, x_1 + 0.2\, x_2 + 0.3\, x_3 + 0.2\, x_4 + 0.2\, x_5 \leq 85$$

$$0.2\, x_5 \leq 20$$

$$x_4 + x_5 \leq 200$$

$$x_1, x_2, x_3 \quad \text{all} \quad \leq 100$$

and all variables non-negative

The optimal production plan based on the above model is to run 37.5, 100, 58.333, and 100 M barrels/week of the four crudes through the fuels process and 100 M barrels/week of crude 4 through the lube process. The corresponding profit is \$67,883/week.

In actuality, the yields, which are the coefficients of the first four constraints, will be subject to fluctuations and the values in Table 1 will be just mean values. Suppose each yield can be expected to vary by $\pm\alpha_{ij}$ and these bounds are collected into the array α

$$\alpha = \begin{bmatrix} 0.07 & 0.07 & 0.03 & 0.05 & 0.05 \\ 0.015 & 0.015 & 0.03 & 0.03 & 0.01 \\ 0.01 & 0.015 & 0.03 & 0.015 & 0.015 \\ 0 & 0 & 0 & 0 & 0.015 \end{bmatrix}$$

Suppose further that once agreement has been made to purchase x barrels/week of crude then any future quantities above or below this contracted level will only be obtainable if an option is added to the contract at costs of 15, 30, 10.5, and 22.5 \$/M barrel of crudes 1 through 4, respectively. The problem thus is to produce an optimal plan for the refinery which takes into account the uncertainties in the scheduling model. As stated, the problem is one with independent variations, Case 1, and thus requires solution of problem (10) with $\delta \bar{a}_{ij} = \alpha_{ij}$. The optimal running plan x remains unchanged; however, $z^+ = 0$ and

$$z^- = (24.594, 0, 5.3251, 0, 6.9767) \text{ M barrel/week}$$

This indicates that because of the uncertainty in the yields provisions should be made now to decrease x by up to z^- M barrel/week in the future. Note that particularly in the case of crude 1, this is a substantial correction considering that the variability in the yields is only about $\pm 10\%$.

To further illustrate our methodology, observe that from the conservation law, it follows that the yields are not independent; rather, the total yield of all products obtained per barrel of a given crude is limited to some constant, say 1. Hence, the variations in the coefficients of each column of the first four constraints are related through an equation of the type,

$$\sum_{i=1}^{4} \eta_{ij}\, \delta a_{ij} = 0 \qquad j = 1, \ldots, 5$$

$$\text{where } \eta_{ij} = \begin{cases} 0 & i = 4 \text{ and } j = 1, \ldots, 4 \\ 1 & \text{otherwise} \end{cases}$$

We thus have a case of column dependence. The perturbations δa^*_{ij} required in formulating problem (10) are obtained by using Proposition 2. The result is

$$(\delta a^*_{ij}) = \begin{bmatrix} 0.025 & 0.03 & 0.03 & 0.045 & 0.04 \\ 0.015 & 0.015 & 0.03 & 0.03 & 0.01 \\ 0.01 & 0.015 & 0.03 & 0.015 & 0.015 \\ 0 & 0 & 0 & 0 & 0.015 \end{bmatrix}$$

Upon solving problem (10), the optimal running plan x again remains unchanged; $z^+ = 0$, and

$$z^- = (9.9334, 0, 14.877, 0, 6.9767) \text{ M barrels/week}$$

As expected, the length of the correction vector z^- in the dependent case is less than that obtained in the independent case. The independent case is always more conservative, that is, requires a larger overall correction provision.

Finally, to illustrate the computations required in the row dependent case we consider the dual (Beightler and Wilde, 1967) of our example,

Minimize

$$170\, y_1 + 85\, y_2 + 85\, y_3 + 20\, y_4 + 100\, y_5 + 100\, y_6 + 100\, y_7 + 200\, y_8$$

Subject to

$$\begin{bmatrix} 0.6 & 0.2 & 0.1 & 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0.2 & 0.2 & 0 & 0 & 1 & 0 & 0 \\ 0.3 & 0.3 & 0.3 & 0 & 0 & 0 & 1 & 0 \\ 0.4 & 0.3 & 0.2 & 0 & 0 & 0 & 0 & 1 \\ 0.4 & 0.1 & 0.2 & 0.2 & 0 & 0 & 0 & 1 \end{bmatrix} y \geq \begin{bmatrix} 100 \\ 200 \\ 70 \\ 150 \\ 250 \end{bmatrix}$$

and $y \geq 0$

Because the above array is the transpose of the primal matrix of constraint coefficients, the column dependency restrictions become row dependency restrictions,

$$\sum_{j=1}^{4} \mu_{ij}\, \delta a_{ij} = 0 \qquad i = 1, \ldots, 5$$

$$\text{where } \mu_{ij} = \begin{cases} 0 & j = 4 \text{ and } i = 1, \ldots, 4 \\ 1 & \text{otherwise} \end{cases}$$

The unperturbed optimal solution is

$$y^* = (133.33, 100.0, 0, 600.0, 0, 113.33, 0, 66.67)$$

and the dual objective value equals that obtained in maximizing the primal, as required. Assuming a flexibility cost for each variable of 10% of its objective function coefficient, the optimal policy y^* remains unchanged; however, $z^- = 0$ and

$$z^+ = (0, 4.35, 7.28, 39.41, 0.098, 0, 0.293, 0)$$

Because the ordering of the ratios $(y_j^* + z_j^+ - z_j^-)/\mu_{ij}$ does not change after the subproblem resulting from the first cut is solved; the iterative procedure terminates after just one pass. Hence, only one perturbed equation set needed to be considered in obtaining the optimal policy with flexibility, namely,

$$(A + \delta A) = \begin{bmatrix} 0.575 & 0.215 & 0.11 & 0 & 1 & 0 & 0 & 0 \\ 0.470 & 0.215 & 0.215 & 0 & 0 & 1 & 0 & 0 \\ 0.270 & 0.3 & 0.33 & 0 & 0 & 0 & 1 & 0 \\ 0.355 & 0.33 & 0.215 & 0 & 0 & 0 & 0 & 1 \\ 0.390 & 0.11 & 0.215 & 0.185 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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NOTATION

a_{ij} = coefficients of the linear inequality constraints
c_i = cost coefficients
b_i = constraint right-hand side
t_i = dummy variable
x_j = primal variables
y_j = dummy variable
z = correction variables
m_k = the kth element of an ordered sequence
A = matrix of constraint coefficients
B = weighting matrix
C = set of indices of columns which have a dependency in the coefficients of that column
$E(\,)$ = expected value operator

$F(\ )$ = the perturbed feasible region resulting from the perturbation in the parenthesis
F^* = the permanently feasible region
N = number of primal variables
M = number of constraints
P = the set of all permissible perturbations in the model coefficients
R = set of indices of rows which have a dependency in the coefficients of that row
α_{ij} = the bounds on the perturbations in the a_{ij}
β_i = the bounds in the perturbations in the b_i
γ_i = a probability
μ_{ij} = coefficients of a row dependency equation
η_{ij} = coefficients of a column dependency equation
δ = a perturbation of the quantity following it

Superscripts

an element in a sequence

Subscripts

a component of a matrix

Mathematical Symbols

ϵ = a member of
$\neq$ = not equal to
ϕ = the empty set
$\cap$ = set intersection
$\cup$ = set union
$\subset$ = set inclusion
$:$ = such that

LITERATURE CITED

Beightler, C. S., and D. J. Wilde, "Sensitivity Analysis Gives Better Insight Into Linear Programming," *Hydrocarbon Proc. Petrol. Ref.*, **44**, 111 (1965).

Charnes, A., and W. W. Cooper, "Chance-Constrained Programming," *Management Sci.*, **6**, 73 (1959).

———., "Deterministic Equivalents for Optimizing and Satisficing Under Chance Constraints," *Operations Res.*, **11**, 18 (1963).

Charnes, A., M. J. L. Kirby, and W. M. Raike, "Zero-zero Chance Constrained Games" in *Proc. 4th Intern. Conf. Opns. Res.*, D. B. Hertz and J. Meleze (eds.), Boston (1966).

Dantzig, G. B., "Linear Programming Under Uncertainty," *Man. Sci.*, **1**, 197 (1955).

Vajda, S., *Probabilistic Programming*, Academic Press, New York (1972).

Walkup, D. W., and R. Wets, "Stochastic Programs with Recourse," *Proc. Princeton Symp. Math. Progr.*, H. W. Kuhn (ed.), Princeton Univ. Press, New Jersey (1970).

Wilde, D. J., and C. S. Beightler, *Foundations of Optimization*, pp. 359-364, Prentice-Hall, Englewood Cliffs, N. J. (1967).

Zangwill, W. I., *Nonlinear Programming*, Ch. 14, Prentice-Hall, Englewood Cliffs, N. J. (1969).

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Flexible Solutions to Linear Programs under Uncertainty: Equality Constraints

This paper presents a framework which allows uncertainties in the matrix elements of an equality constrained linear program to be taken into account without requiring detailed knowledge of the statistical characteristics of these uncertainties. The results are derived using the model of the linear program with flexibility previously introduced for the inequality constrained case. However, because a feasible region common to all perturbed constraint sets does not exist in the equality constrained case, a flexibility set which intersects all perturbed sets individually rather than jointly is defined. The flexibility set is constructed by identifying a finite subset of all perturbed constraint sets which need to be investigated. Three cases for the equality constrained problem are considered: independent variations in the array elements, column dependent variations, and row dependent variations. In each case the problem is solved using a possibly large but decomposable linear program. In the first two cases, this program needs to be solved only once; while in the row dependent case an iterative but finite solution procedure is required.

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SCOPE

In the preceding paper (Friedman and Reklaitis, 1974) a model and solution procedure were presented which would allow uncertainties in the array and right-hand side coefficients of an inequality constrained linear program to be taken into account. The model, called the *linear program with flexibility*, was based on the assumption that the model builder will know:

1. A mean, or "best estimate," value for each matrix coefficient,
2. The intervals over which variations in each matrix coefficient are assumed to be uniformly distributed,
3. The coefficients of any linear relationships between the matrix elements which impose restrictions on the possible variations, and